3.5 CLAIRAUT'S EQUATION

An equation of the form $y = px + f(p)$ ...(1)

is known as Clairaut's equation.

Differentiating (1) w.r.t. x, we get

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx} \quad \text{or} \quad [x + f'(p)]\frac{dp}{dx} = 0$$

Discarding the factor $[x + f'(p)]$, we have $\frac{dp}{dx} = 0$

Integrating, $p = c$

Putting $p = c$ in (1), the required solution is $p = cx + f(c)$

Thus, the solution of Clairaut's equation is obtained by writing c for p.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve $(y - px)(p - 1) = p$.*

Sol. The given equation can be written as

$$y - px = \frac{p}{p-1} \quad \text{or} \quad y = px + \frac{p}{p-1}$$

This is of Clairaut's form. Hence putting c for p, the solution is $y = cx + \frac{c}{c-1}$.

Note. Many differential equations can be reduced to Clairaut's form by suitably changing the variables.

Example 2. *Solve* $e^{4x}(p-1)+e^{2y}p^2=0$.

Sol. [In problems involving e^{lx} and e^{my}, put $X=e^{kx}$ and $Y=e^{ky}$, where k is the H.C.F. of l and m].

Put $\quad X=e^{2x}$ and $Y=e^{2y}$

so that $\quad dX=2e^{2x}\,dx$ and $dY=2e^{2y}\,dy$

$$\therefore \quad p=\frac{dy}{dx}=\frac{e^{2x}}{e^{2y}}\frac{dY}{dX}=\frac{X}{Y}P, \quad \text{where } P=\frac{dY}{dX}$$

The given equation becomes $X^2\left(\frac{X}{Y}P-1\right)+Y\cdot\frac{X^2}{Y^2}P^2=0$.

or $\quad XP-Y+P^2=0$ or $Y=PX+P^2$ which is of Clairaut's form.

$\therefore$ Its solution is $Y=cX+c^2$ and hence $e^{2y}=ce^{2x}+c^2$.

Example 3. *Solve* $(px-y)(py+x)=2p$.

Sol. Put $\quad X=x^2$ and $Y=y^2$ so that $dX=2x\,dx$ and $dY=2y\,dy$

$$\therefore \quad p=\frac{dy}{dx}=\frac{x}{y}\cdot\frac{dY}{dX}=\frac{\sqrt{X}}{\sqrt{Y}}P, \quad \text{where } P=\frac{dY}{dX}$$

The given equation becomes

$$\left(\frac{\sqrt{X}}{\sqrt{Y}}P\cdot\sqrt{X}-\sqrt{Y}\right)\left(\frac{\sqrt{X}}{\sqrt{Y}}P\cdot\sqrt{Y}+\sqrt{X}\right)=2\frac{\sqrt{X}}{\sqrt{Y}}P$$

or $$(PX-Y)(P+1)=2P \quad \text{or} \quad PX-Y=\frac{2P}{P+1}$$

or $$Y=PX-\frac{2P}{P+1} \quad \text{which is of Clairaut's form.}$$

$\therefore$ Its solution is $Y=cX-\dfrac{2c}{c+1}$ and hence $y^2=cx^2-\dfrac{2c}{c+1}$.

TEST YOUR KNOWLEDGE

Solve the following equations:

1. $y=xp+\dfrac{a}{p}$
2. $y=px+\sqrt{a^2p^2+b^2}$
3. $\sin px\cos y=\cos px\sin y+p$
4. $xp^2-yp+a=0$
5. $(x-a)p^2+(x-y)p-y=0$
6. $p=\log(px-y)$
7. $p=\sin(y-px)$
8. $p^2(x^2-1)-2pxy+y^2-1=0$
9. $e^{3x}(p-1)+p^3e^{2y}=0$
10. $x^2(y-px)=yp^2$
11. $(y+px)^2=x^2p$.

Answers

1. $y=cx+\dfrac{a}{c}$
2. $y=cx+\sqrt{a^2c^2+b^2}$
3. $y=cx-\sin^{-1}c$
4. $y=cx+\dfrac{a}{c}$
5. $y=cx-\dfrac{ac^2}{c+1}$
6. $y=cx-e^c$
7. $y=cx+\sin^{-1}c$
8. $(y-cx)^2=1+c^2$
9. $e^y=ce^x+c^2$
10. $y^2=cx^2+c^2$ [**Hint.** Put $x^2=X$, $y^2=Y$]
11. $xy=cx-c^2$. [**Hint.** Put $xy=v$]

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